BIRZEIT UNIVERSITY

Mathematics Department

Math 332
Dr. Alaeddin Elayyan

Third Exam 1st. Semester 2018/2019
Student name: .... ID no.: .... sec........

[1](18 points)
(a) (8 points) Use separation of variables to derive the solution of

$$u_{tt} = 4u_{xx} \qquad 0 \le x \le \pi\ ,\quad t \ge 0$$
$$u(0,t) = 0$$
$$u(\pi,t) = 0$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = 0$$

$u(x,t) = X(x)T(t) \Rightarrow XT'' = 4X''T$

$\Rightarrow \frac{X''}{X} = \frac{T''}{4T} = -\lambda$

2 $X'' + \lambda X = 0$ & $T'' + 4\lambda T = 0$

$X(0) = 0,\ X(\pi) = 0$

2 $\lambda = n^2,\ X_n = \sin nx \quad ,\ T'' + 4n^2T = 0$

$u(x,t) = \sum_{n=1}^{\infty} (A_n \sin 2nt + B_n \cos 2nt) \sin nx$

2 $u_t(x,0) = 0 \Rightarrow A_n = 0$

$u(x,t) = \sum_{n=1}^{\infty} B_n \cos 2nt \sin nx$

$u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx = f(x) \Rightarrow B_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$

2 using the $\sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$

$A = nx,\ B = 2nt$

$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n[\sin(nx + 2nt) + \sin(nx - 2nt)]$

$= \frac{1}{2} \sum_{n=1}^{\infty} B_n \sin n(x+2t) + B_n \sin n(x-2t)$

$= \frac{1}{2}\left(f(x+2t) + f(x-2t)\right)$

(b) (8 points) Derive the solution of the following problem using Fourier transform

$u_{tt} = 4u_{xx} \qquad 0 \le x \le \pi\,, \quad t \ge 0$

$u(0,t) = 0$

$u(\pi,t) = 0$

$u(x,0) = 0$

$u_t(x,0) = g(x)$

$$\phi(u) = \hat{U} = \int_{-\infty}^{\infty} u(x,t)\, e^{-i w x}\, dx$$

Let $\hat{U} = \phi(u)$

$$\Rightarrow \hat{U}_{tt} = -4w^2 \hat{U}$$

$$\Rightarrow \hat{U}_{tt} + 4w^2 \hat{U} = 0$$

2

$$\hat{U} = A \sin 2wt + B \cos 2wt$$

$u(x,0) = 0 \Rightarrow \hat{U}(w,0) = 0 \Rightarrow B = 0$

$$\hat{U} = A \sin 2wt$$

$$\hat{U}_t = 2wA \cos 2wt \Big|_{t=0} = \hat{g}(w)$$

$$\Rightarrow A = \frac{\hat{g}(w)}{2w}$$

2

$$\Rightarrow \hat{U} = \frac{\hat{g}\sin 2wt}{2w} = \frac{1}{4}\hat{g}(w)\frac{2\sin 2wt}{w}$$

$$= \frac{1}{4}\hat{g}(w)\,\phi\left(\begin{cases}1 & |x|<2t \\ 0 & x>2t\end{cases}\right)$$

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$$u(x,t) = \frac{1}{4} g(x) * \begin{cases} 1, & |x|<2t \\ 0, & |x|>2t \end{cases}$$

$$= \frac{1}{4}\int_{-2t}^{2t} g(x-u)\,du = \frac{1}{4}\int_{x-2t}^{x+2t} g(s)\,ds$$

$s = x - u$

(c) (2 point) Derive the solution of

$u_{tt} = 4u_{xx} \qquad 0 \le x \le \pi\,, \quad t \ge 0$

$u(0,t) = 0$

$u(\pi,t) = 0$

$u(x,0) = f(x)$

$u_t(x,0) = g(x)$

Let $u = w + v$

where w satisfies problem a

& v " b

then $u_{tt} = w_{tt} + v_{tt} = 4w_{xx} + 4v_{xx} = 4u_{xx}$

& $u(0,t) = w(0,t) + v(0,t) = 0 + 0 = 0$

$$\hat{u}(x,t) = \frac{1}{2}\, \hat{g}(\omega)\, \frac{\sin 2\omega t}{\omega}$$

$$= \frac{i}{2}\, \frac{\hat{g}(\omega)\sin 2\omega t}{i\omega}$$

$$= \frac{i}{2}\, \sin 2\omega t\, \frac{\hat{g}(\omega)}{i\omega}$$

$$= \frac{i}{2}\, \sin 2\omega t\, \hat{f}(\omega)$$

$$= \frac{i}{2}\, \frac{f(x+2t) - f(x-2t)}{2i}$$

$$= \frac{1}{4}\left(\int_{-\infty}^{x+2t} g(u)\,du - \int_{-\infty}^{x-2t} g(u)\,du \right)$$

$$= \frac{1}{4} \int_{x-2t}^{x+2t} g(u)\,du$$

[2] (18 points)

(a) (10 points) For the wave equation in problem (1) if

$$f(x) = \begin{cases} x & , \quad |x| \le 1 \\ 0 & , \quad o.w \end{cases} \qquad , \qquad g(x) = \begin{cases} 3 & 0 < x \le 2 \\ 0 & o.w \end{cases}$$

Find $u(x,2)$ and graph it.

$$u(x,t) = \frac{f(x+2t)+f(x-2t)}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} g(s)\,ds$$

$$u_f = \frac{f(x-2t)}{2} - \frac{1}{4}\int_{0}^{x-2t} g(s)\,ds$$

$\underline{t=2}$

$$= \frac{f(x-4)}{2} - \frac{1}{4}\int_{0}^{x-4} g(s)\,ds$$

$$\frac{f(x-4)}{2} = \begin{cases} \frac{x-4}{2} & , \quad 3<x<5 \quad (-1 \le x-4 \le 1) \\ 0 & , \quad o.w \end{cases}$$

$$-\frac{1}{4}\int_{0}^{x-4} g(s)\,ds = \begin{cases} 0 & , \quad x<4 \\ -\frac{1}{4}\int_0^{x-4} 3\,ds = -\frac{3}{4}(x-4) & \quad 4<x<6 \\ -\frac{3}{2} & \quad x>6 \end{cases}$$

$$u_f = \begin{cases} 0 & x<3 \\ \frac{x-4}{2} & 3<x<4 \\ \frac{x-4}{2} - \frac{3}{4}(x-4) & 4<x<5 \\ -\frac{3}{4}(x-4) & 5<x<6 \\ -\frac{3}{2} & x>6 \end{cases}$$

3

[2] (18 points)

(a) (10 points) For the wave equation in problem (1) if

$$f(x) = \begin{cases} x & , \quad |x| \le 1 \\ 0 & , \quad o.w \end{cases} \quad , \qquad g(x) = \begin{cases} 3 & 0 < x \le 2 \\ 0 & o.w \end{cases}$$

Find $u(x, 2)$ and graph it.

$$u_b = \frac{f(x+4)}{2} + \frac{1}{4}\int_0^{x+4} g(s)\,ds$$

$$\frac{f(x+4)}{2} = \begin{cases} \frac{x+4}{2} & , -5 \le x \le -3 \\ 0 & , o.w \end{cases}$$

$$\frac{1}{4}\int_0^{x+4} g(s)\,ds = \begin{cases} 0 & x \le -\cancel{4} \\ \frac{1}{4}\int_0^{x+4} 3\,ds = \frac{3}{4}(x+4) & , -\cancel{4} < x \le -2 \\ \frac{3}{2} & x > -2 \end{cases}$$

$$u_b = \begin{cases} 0 & x \le -\cancel{5} \\ \frac{x+4}{2} & -5 < x \le -\cancel{4} \\ \frac{x+4}{2} + \frac{3}{4}(x+4) & -4 \le x \le -3 \\ \frac{3}{4}(x+4) & -3 < x \le -2 \\ \frac{3}{2} & x > -2 \end{cases}$$

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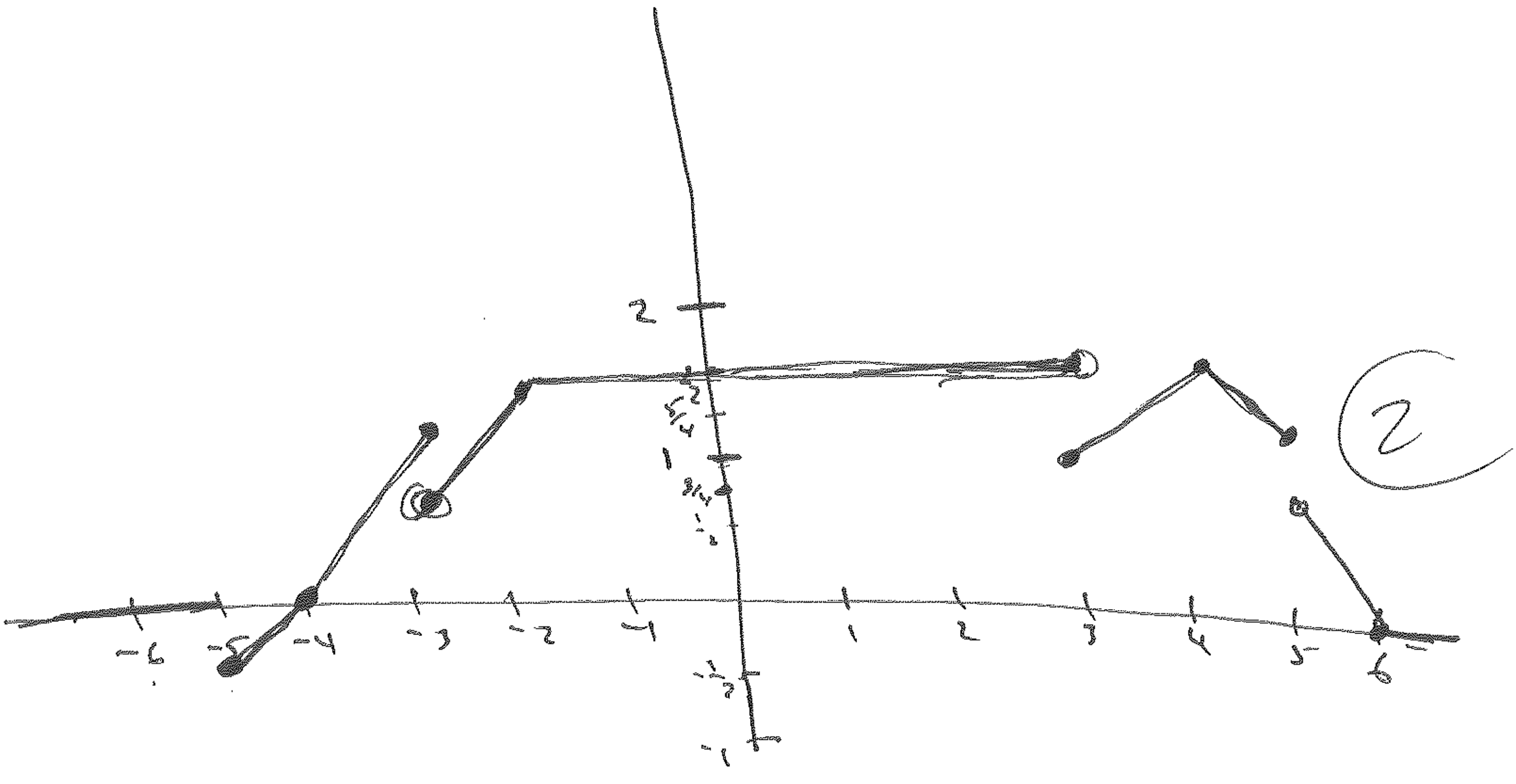
$$u(x,t) = u_f + u_b$$

$$= \begin{cases} 0 & x \le -5 \\ \dfrac{x+4}{2} & -5 \le x \le -4 \\ \dfrac{x+4}{2} + \dfrac{3}{4}(x+4) & -4 \le x < -3 \\ \dfrac{3}{4}(x+4) & -3 \le x < -2 \\ \dfrac{3}{2} & -2 \le x < 3 \\ \dfrac{x-4}{2} + \dfrac{3}{2} & 3 \le x < 4 \\ \dfrac{x-4}{2} + \dfrac{3}{2} - \dfrac{3}{4}(x-4) & 4 \le x \le 5 \\ -\dfrac{3}{4}(x-4) + \dfrac{3}{2} & 5 \le x \le 6 \\ 0 & x > 6 \end{cases}$$

2



2

b)(8 points) Find the general solution of the given PDE using the change of variables

$$\mu = x - ct, \qquad \gamma = t,$$

$u_t(x,t) + cu_x(x,t) = 0,$

$u(x,0) = f(x)$

Soln

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \mu}\frac{\partial \mu}{\partial t} + \frac{\partial u}{\partial \gamma}\frac{\partial \gamma}{\partial t}$$

$$= u_\mu \cdot (-c) + u_\gamma \cdot 1$$

2 $$= -c\,u_\mu + u_\gamma$$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \mu}\frac{\partial \mu}{\partial x} + \frac{\partial u}{\partial \gamma}\frac{\partial \gamma}{\partial x}$$

$$= u_\mu (1) + u_\mu (0)$$

2 $$= u_\mu$$

$$u_t + c u_x = -c u_\mu + u_\gamma + c u_\mu$$

1 $$= u_\gamma = 0$$

1 $$u(\mu,\gamma) = \int u_\gamma \, d\gamma + h(\mu)$$

$$= \int 0 \, d\gamma + h(\mu)$$

$$= h(\mu)$$

$$u(x,t) = h(x-ct)$$

1 $$u(x,0) = h(x) = f(x)$$

1 $$u(x,t) = f(x-ct).$$